

Augmented superfield approach to unique nilpotent symmetries for complex scalar fields in QED

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Received: 15 April 2006 / Revised version: 9 June 2006 /

Published online: 24 October 2006 – © Springer-Verlag / Società Italiana di Fisica 2006

Abstract. The derivation of the exact and unique nilpotent Becchi–Rouet–Stora–Tyutin (BRST) and anti-BRST symmetries for the matter fields, present in any arbitrary interacting gauge theory, has been a long-standing problem in the framework of the superfield approach to the BRST formalism. These nilpotent symmetry transformations are deduced for the four (3+1)-dimensional (4D) complex scalar fields, coupled to the $U(1)$ gauge field, in the framework of an augmented superfield formalism. This interacting gauge theory (i.e. QED) is considered on a six (4,2)-dimensional supermanifold parametrized by four *even* spacetime coordinates and a couple of *odd* elements of the Grassmann algebra. In addition to the horizontality condition (that is responsible for the derivation of the exact nilpotent symmetries for the gauge field and the (anti-) ghost fields), a new restriction on the supermanifold, owing its origin to the (super) covariant derivatives, has been invoked for the derivation of the exact nilpotent symmetry transformations for the matter fields. The geometrical interpretations for all the above nilpotent symmetries are discussed, too.

PACS. 11.15.-q; 12.20.-m; 03.70.+k

1 Introduction

The application of the Becchi–Rouet–Stora–Tyutin (BRST) formalism to gauge theories (endowed with the first-class constraints in the language of Dirac’s prescription for the classification scheme [1,2]) stands on a firm ground because

- (i) it provides the covariant canonical quantization of these theories [3–7],
- (ii) the unitarity and the “quantum” gauge (i.e. BRST) invariance are respected *together* at any arbitrary order of perturbative computations related to a given physical process (see, e.g., [3,4,8]),
- (iii) its salient features are intimately connected with the mathematical aspects of differential geometry and cohomology (see, e.g., [9–14]), and
- (iv) it has deep relations with some of the key ideas associated with the supersymmetry.

In our present investigation, we shall touch upon some of the issues related with the geometrical aspects of the BRST formalism, applied to an interacting $U(1)$ gauge theory (i.e. QED), in the framework of the superfield formalism [15–32].

The *usual* superfield approach to BRST formalism [15–24] provides the geometrical interpretation for the con-

served and nilpotent (anti-) BRST charges (and the corresponding nilpotent and anticommuting (anti-) BRST symmetries they generate) for the Lagrangian density of a given 1-form (non-) Abelian gauge theory defined on the four (3+1)-dimensional (4D) spacetime manifold. The key idea in this formulation is to consider the original 4D 1-form (non-) Abelian gauge theory on a six (4,2)-dimensional supermanifold parametrized by the four spacetime (even) coordinates x^μ ($\mu = 0, 1, 2, 3$) and a couple of Grassmannian (odd) variables θ and $\bar{\theta}$ (with $\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$). One constructs, especially for the 4D 1-form non-Abelian gauge theory, the super curvature 2-form $\tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} + \tilde{A}^{(1)} \wedge \tilde{A}^{(1)}$ with the help of the super exterior derivative \tilde{d} (with $\tilde{d}^2 = 0$) and the super 1-form connection $\tilde{A}^{(1)}$. This is subsequently equated, due to the so-called horizontality condition¹ [15–24], to the ordinary 2-form curvature $F^{(2)} = dA^{(1)} + A^{(1)} \wedge A^{(1)}$ constructed with the help of the ordinary

¹ This condition has also been applied to the 2-form ($A^{(2)} = \frac{1}{2!}(dx^\mu \wedge dx^\nu)B_{\mu\nu}$) Abelian gauge theory where the 3-form super curvature $\tilde{F}^{(3)} = \tilde{d}\tilde{A}^{(2)}$, defined on the six (4,2)-dimensional supermanifold, is equated to the ordinary 3-form $F^{(3)} = dA^{(2)}$ curvature, defined on the 4D ordinary Minkowskian spacetime manifold. As expected, this restriction leads to the derivation of nilpotent (anti-) BRST symmetry transformations for the 2-form gauge field and the associated (anti-) ghost fields of the theory [24].

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exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) and the 1-form ordinary connection $A^{(1)}$. The above restriction is referred to as the soul-flatness condition in [7] which amounts to setting equal to zero all the Grassmannian components of the second-rank (anti-) symmetric curvature tensor that is required in the definition of the 2-form supercurvature $\tilde{F}^{(2)}$ on the six (4, 2)-dimensional supermanifold.

The covariant reduction of the six (4, 2)-dimensional supercurvature $\tilde{F}^{(2)}$ to the 4D ordinary curvature $F^{(2)}$ in the horizontality restriction (i.e. $\tilde{F}^{(2)} = F^{(2)}$) leads to

- (i) the derivation of the nilpotent (anti-) BRST symmetry transformations for the gauge field and the (anti-) ghost fields of the 1-form non-Abelian gauge theory,
- (ii) the geometrical interpretation for the (anti-) BRST charges as the translation generators along the Grassmannian directions of the supermanifold,
- (iii) the geometrical meaning of the nilpotency property which is found to be encoded in a couple of successive translations (i.e. $(\partial/\partial\theta)^2 = (\partial/\partial\bar{\theta})^2 = 0$) along any particular Grassmannian direction (i.e. θ or $\bar{\theta}$) of the supermanifold, and
- (iv) the geometrical interpretation for the anticommutativity property of the BRST and anti-BRST charges that are found to be captured by the relation $(\partial/\partial\theta)(\partial/\partial\bar{\theta}) + (\partial/\partial\bar{\theta})(\partial/\partial\theta) = 0$.

It should be noted, however, that these beautiful connections between the geometrical objects on the supermanifold and the (anti-) BRST symmetries (as well as the corresponding generators) for the ordinary fields on the ordinary manifold remain confined *only* to the gauge and (anti-) ghost fields of an *interacting* gauge theory. This *usual* superfield formalism does not shed any light on the nilpotent and anticommuting (anti-) BRST symmetry transformations associated with the matter fields of an interacting (non-) Abelian gauge theory. It has been a long-standing problem to find these nilpotent symmetries for the matter fields in the framework of superfield formalism.

In a recent set of papers [25–32], the above usual superfield formalism (endowed with the horizontality condition *alone*) has been consistently extended to include, in addition, the invariance of conserved quantities on the supermanifold (see, e.g., [31] for details). It has been also established in [25–32] that the invariance of the conserved (super) matter currents on the (super) spacetime manifolds leads to the derivation of the *consistent* set of nilpotent symmetry transformations for the matter fields of a given four dimensional interacting 1-form (non-) Abelian gauge theory (see, e.g., [25–30]). The salient features of the above extensions (and, in some sense, generalizations) of the usual superfield formulation are

- (i) the geometrical interpretations for the nilpotent and anticommuting (anti-) BRST symmetry transformations (and their corresponding generators) remain intact for all the fields (including the matter fields) of the interacting gauge theory,

- (ii) there is a mutual consistency and conformity between the additional restrictions imposed on the supermanifold and the usual restriction due to the horizontality condition, and
- (iii) it has been found that these derivations of the nilpotent symmetries (especially for the matter fields) are *not* unique mathematically.

In a very recent paper [32], the mathematical uniqueness has been shown for the derivation of the off-shell nilpotent and anticommuting (anti-) BRST symmetry transformations for the Dirac fields coupled to the $U(1)$ gauge field.

The purpose of our present paper is to show that the ideas of the augmented superfield formalism, proposed in [32], can be extended to derive the off-shell nilpotent and anticommuting (anti-) BRST symmetry transformations for *all* the fields of an interacting four (3 + 1)-dimensional (4D) $U(1)$ gauge theory where there is an interaction between the charged complex scalar fields and the photon (i.e. QED). We demonstrate that there is a mutual consistency, conformity and complementarity between

- (i) the horizontality condition, and
- (ii) a *new* restriction on the six (4, 2)-dimensional supermanifold on which our present 4D interacting gauge theory is considered.

The latter restriction owes its origin to the (super) covariant derivatives on the (super) spacetime manifolds and leads to the exact and unique derivation of the nilpotent and anticommuting (anti-) BRST symmetry transformations for the matter (complex scalar) fields. As is well known [15–24], the former restriction too depends on the (super) covariant derivatives on the (super) spacetime manifolds in a *different* way (than the latter) and leads to the derivation of the nilpotent and anticommuting (anti-) BRST symmetry transformations for the gauge and (anti-) ghost fields in an exact and unique fashion. We show, in an explicit manner, that *only* the *gauge-invariant* versions (cf. (13) and (37) below) of the *new* restriction on the supermanifold lead to the exact derivation of the nilpotent symmetry transformations for the matter fields of the present QED. The covariant versions (cf. (A.1) and associated footnote in the appendix) of the new restriction lead to physically unacceptable solutions.

Our present investigation is interesting as well as essential primarily on three accounts.

First, it is the generalization of our previous idea for the derivation of the unique nilpotent symmetries associated with the Dirac fields in QED [32] to a more complicated system of QED where the charged complex scalar fields interact with photon. This generalization is an important step towards putting our proposed idea of a new restriction (on the six (4, 2)-dimensional supermanifold [32]) onto a firmer footing for a *new* interacting gauge system where the conserved Noether current (that couples to the $U(1)$ gauge field) contains the $U(1)$ gauge field itself. It will be noted that, for QED with the Dirac fields, the conserved current (that couples to the $U(1)$ gauge field) contains only the fermionic Dirac fields (and no $U(1)$ gauge field).

Second, our present example of the interacting gauge theory (QED) is more interesting, in some sense, than its counterpart with the Dirac fields because the phenomena of spontaneous symmetry breaking, Higgs mechanism, Goldstone theorem, etc. are associated with our present system which are not found to exist for the latter system of interacting $U(1)$ gauge theory.

Finally, our present system of a gauge field theory allows for the inclusion of a quartic renormalizable potential for the matter fields in the Lagrangian density (cf. (1) and (3) below) which is $U(1)$ gauge (as well as (anti-) BRST) invariant. Such a kind of $U(1)$ gauge (as well as (anti-) BRST) invariant potential, for the matter fields, does not exist for the QED with Dirac fields.

The contents of our present paper are organized as follows. To set up the notation and conventions for the main body of the text, in Sect. 2, we provide a brief synopsis of the off-shell nilpotent (anti-) BRST symmetries for the 4D interacting $U(1)$ gauge theory (QED) in the Lagrangian formulation where the gauge field A_μ couples to the Noether conserved current constructed by the complex scalar fields and A_μ itself. For the sake of this paper being self-contained, Sect. 3 deals with the derivation of the above nilpotent symmetries for the gauge- and (anti-) ghost fields in the framework of usual superfield formulation where the horizontality condition on the six (4, 2)-dimensional supermanifold plays a very decisive role [15–24]. The central results of our paper are accumulated in Sect. 4 where we derive the off-shell nilpotent symmetries for the complex scalar fields by exploiting a *gauge-invariant* restriction on the supermanifold. A very important point, connected with this section, is discussed in an appendix of our present paper (cf. Appendix A). Finally, we summarize our key results, make some concluding remarks and point out a few promising future directions for further investigations in Sect. 5.

2 Nilpotent (anti-) BRST symmetries: Lagrangian formulation

To recapitulate the key points connected with the local, covariant, continuous, anticommuting and off-shell nilpotent (anti-) BRST symmetries, we focus on the Lagrangian density of an *interacting* four (3+1)-dimensional² (4D) $U(1)$ gauge theory which describes a dynamically closed system of charged complex scalar fields and the photon (i.e.

² We adopt here the conventions and notation such that the 4D flat Minkowski metric is: $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ and $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu = (\partial_0)^2 - (\partial_i)^2$, $F_{0i} = \partial_0 A_i - \partial_i A_0 = E_i \equiv \mathbf{E}$, $F_{ij} = \epsilon_{ijk} B_k$, $B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} \equiv \mathbf{B}$, $(\partial \cdot A) = \partial_0 A_0 - \partial_i A_i$ where \mathbf{E} and \mathbf{B} are the electric and magnetic fields, respectively and ϵ_{ijk} is the totally antisymmetric Levi-Civita tensor defined on the 3D space sub-manifold of the 4D spacetime manifold. Here the Greek indices, $\mu, \nu, \dots = 0, 1, 2, 3$, correspond to the spacetime directions, and the Latin indices, $i, j, k, \dots = 1, 2, 3$, stand only for the space directions on the Minkowski spacetime manifold.

QED). The (anti-) BRST invariant version of the above Lagrangian, in the Feynman gauge, is [3–7]

$$\begin{aligned} \mathcal{L}_B &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - V(\phi^* \phi) + B(\partial \cdot A) \\ &\quad + \frac{1}{2} B^2 - i \partial_\mu \bar{C} \partial^\mu C, \\ &\equiv \frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2) + (D_\mu \phi)^* D^\mu \phi - V(\phi^* \phi) + B(\partial \cdot A) \\ &\quad + \frac{1}{2} B^2 - i \partial_\mu \bar{C} \partial^\mu C, \end{aligned} \quad (1)$$

where $V(\phi^* \phi)$ is the potential describing the interaction between the complex scalar fields ϕ and ϕ^* and the covariant derivatives on these fields, with the electric charge e , are

$$D_\mu \phi = \partial_\mu \phi + ie A_\mu \phi, \quad (D_\mu \phi)^* = \partial_\mu \phi^* - ie A_\mu \phi^*. \quad (2)$$

It will be noted that, in general, the potential $V(\phi^* \phi)$ can be chosen to possess a quartic renormalizable interaction term which turns out to be $U(1)$ gauge invariant (see, e.g. [33] for details). The Lagrangian density \mathcal{L}_B includes the gauge fixing term $(\partial \cdot A)$ through the Nakanishi–Lautrup auxiliary field B and the Faddeev–Popov (anti-) ghost fields $(\bar{C}) C$ (with $C^2 = \bar{C}^2 = 0$, $C\bar{C} + \bar{C}C = 0$) are required in the theory to maintain the (anti-) BRST invariance and unitarity *together* at any arbitrary order of perturbative calculations [3, 4, 8]. In the sense of the basic requirements of a canonical field theory, the Lagrangian density \mathcal{L}_B (cf. (1)) describes a dynamically closed system because the quadratic kinetic energy terms and the interaction terms for all the fields ϕ, ϕ^* and A_μ are present in this Lagrangian density in a logical fashion (see, e.g., [33]). It will be noted that the gauge field A_μ couples to the conserved matter current $J_\mu \sim [\phi^* D_\mu \phi - \phi (D_\mu \phi)^*]$ to provide the interaction between

- (i) the $U(1)$ gauge field itself, and
- (ii) the $U(1)$ gauge field and matter fields (i.e. complex scalar fields ϕ as well as ϕ^*).

This statement can be succinctly expressed by re-expressing (1), in terms of the kinetic energy terms for ϕ and ϕ^* , as given here:

$$\begin{aligned} \mathcal{L}_B &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \partial_\mu \phi^* \partial^\mu \phi - ie A_\mu [\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*] \\ &\quad + e^2 A^2 \phi^* \phi - V(\phi^* \phi) + B(\partial \cdot A) + \frac{1}{2} B^2 - i \partial_\mu \bar{C} \partial^\mu C. \end{aligned} \quad (3)$$

The conservation of the matter current J_μ can be easily checked by exploiting the equations of motion $D_\mu D^\mu \phi = -(\partial V / \partial \phi^*)$, $(D_\mu D^\mu \phi)^* = -(\partial V / \partial \phi)$ derived from the Lagrangian densities (1) and/or (3). The above Lagrangian density respects the following off-shell nilpotent ($s_{(a)b}^2 = 0$) and anticommuting ($s_b s_{ab} + s_{ab} s_b = 0$) (anti-) BRST sym-

metry transformations $s_{(a)b}^3$ on the matter fields, gauge field and the (anti-) ghost fields:

$$\begin{aligned} s_b A_\mu &= \partial_\mu C, & s_b C &= 0, & s_b \bar{C} &= iB, & s_b \phi &= -ieC\phi, \\ s_b \phi^* &= +ie\phi^* C, & s_b \mathbf{B} &= 0, & s_b B &= 0, & s_b \mathbf{E} &= 0, \\ s_b(\partial \cdot A) &= \square C, & s_{ab} A_\mu &= \partial_\mu \bar{C}, & s_{ab} \bar{C} &= 0, \\ s_{ab} C &= -iB, & s_{ab} \phi &= -ie\bar{C}\phi, & s_{ab} \phi^* &= +ie\phi^* \bar{C}, \\ s_{ab} \mathbf{B} &= 0, & s_{ab} B &= 0, & s_{ab} \mathbf{E} &= 0, & s_{ab}(\partial \cdot A) &= \square \bar{C}. \end{aligned} \quad (4)$$

The key points to be noted, at this stage, are as follows.

- (i) Under the (anti-) BRST transformations, it is the kinetic energy term $(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu})$ of the gauge field A_μ which remains invariant. This statement is true for any (non-) Abelian gauge theory. For the above $U(1)$ gauge theory, as it turns out, it is the curvature term $F_{\mu\nu}$ (constructed from the operation of the exterior derivative $d = dx^\mu \partial_\mu$ on the 1-form $A^{(1)} = dx^\mu A_\mu$) itself that remains invariant under the (anti-) BRST transformations.
- (ii) In the mathematical language, the (anti-) BRST symmetries owe their origin to the exterior derivative $d = dx^\mu \partial_\mu$ because the curvature term is constructed from it.
- (iii) This observation will be exploited in the next section where (super) exterior derivatives would play very decisive roles in the derivation of the exact nilpotent (anti-) BRST transformations for the gauge and (anti-) ghost fields in the framework of usual superfield formalism.
- (iv) In general, the above transformations can be concisely expressed in terms of the generic field $\Sigma(x)$ and the conserved charges $Q_{(a)b}$, as

$$s_r \Sigma(x) = -i[\Sigma(x), Q_r]_{\pm}, \quad r = b, ab, \quad (5)$$

where the local generic field $\Sigma = A_\mu, C, \bar{C}, B, \phi, \phi^*$ and the (+)– signs, as the subscripts on the square bracket $[\cdot]_{\pm}$, stand for the (anti)commutators for Σ being (fermionic) bosonic in nature. The explicit forms of the conserved and nilpotent charges $Q_{(a)b}$ are not required for our present discussions but can be derived by exploiting the Noether theorem.

³ We follow here the notations and conventions adopted in [5, 6]. In fact, the (anti-) BRST prescription is to replace the local gauge parameter by an anticommuting number η and the (anti-) ghost fields $(\bar{C})C$ which anticommute (i.e. $\eta C + C\eta = 0, \eta \bar{C} + \bar{C}\eta = 0$) and commute (i.e. $\eta B = B\eta, \eta A_\mu = A_\mu \eta$, etc.) with all the fermionic (i.e. $C\bar{C} + \bar{C}C = 0, C^2 = \bar{C}^2 = 0$, etc.) and bosonic (i.e. $B, A_\mu, B^2 \neq 0$, etc.) fields, respectively. In its totality, the nilpotent ($\delta_{(A)B}^2 = 0$) (anti-) BRST transformations $\delta_{(A)B}$ are the product (i.e. $\delta_{(A)B} = \eta s_{(a)b}$) of η and $s_{(a)b}$ where the nilpotency property is carried by $s_{(a)b}$ (with $s_{(a)b}^2 = 0$).

3 Nilpotent symmetries for the gauge-field and (anti-) ghost fields: Usual superfield formalism with horizontality condition

To obtain the off-shell nilpotent symmetry transformations (4) for the $U(1)$ gauge field (A_μ) and anticommuting (anti-) ghost fields $((\bar{C})C)$ in the *usual* superfield formalism, we define the 4D ordinary interacting gauge theory on a six (4, 2)-dimensional supermanifold parametrized by the general superspace coordinate $Z^M = (x^\mu, \theta, \bar{\theta})$ where x^μ ($\mu = 0, 1, 2, 3$) are the four even spacetime coordinates and $\theta, \bar{\theta}$ are a couple of odd elements of a Grassmann algebra. On this supermanifold, one can define a super 1-form connection $\tilde{A}^{(1)} = dZ^M(\tilde{A}_M)$ with the supervector superfield $\tilde{A}_M \equiv (B_\mu(x, \theta, \bar{\theta}), \mathcal{F}(x, \theta, \bar{\theta}), \bar{\mathcal{F}}(x, \theta, \bar{\theta}))$. Here $B_\mu, \mathcal{F}, \bar{\mathcal{F}}$ are the component multiplet superfields, where B_μ is an even superfield and $\mathcal{F}, \bar{\mathcal{F}}$ are the odd superfields [18–21]. These multiplet superfields can be expanded in terms of the basic fields A_μ, C, \bar{C} , auxiliary multiplier field B and some secondary fields as (see, e.g., [18–21])

$$\begin{aligned} B_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta \bar{R}_\mu(x) + \bar{\theta} R_\mu(x) + i\theta \bar{\theta} S_\mu(x), \\ \mathcal{F}(x, \theta, \bar{\theta}) &= C(x) + i\theta \bar{B}(x) + i\bar{\theta} B(x) + i\theta \bar{\theta} s(x), \\ \bar{\mathcal{F}}(x, \theta, \bar{\theta}) &= \bar{C}(x) + i\theta \bar{B}(x) + i\bar{\theta} B(x) + i\theta \bar{\theta} \bar{s}(x). \end{aligned} \quad (6)$$

It is straightforward to note that the local fields $R_\mu(x), \bar{R}_\mu(x), C(x), \bar{C}(x), s(x), \bar{s}(x)$ are fermionic (anticommuting) and $A_\mu(x), S_\mu(x), \mathcal{B}(x), \bar{\mathcal{B}}(x), B(x), \bar{B}(x)$ are bosonic (commuting) in nature. In the above expansion, the bosonic and fermionic degrees of freedom match and, in the limit $\theta, \bar{\theta} \rightarrow 0$, we get back our basic gauge and (anti-) ghost fields A_μ, C, \bar{C} of (1) and/or (3). These requirements are essential for the sanctity of any arbitrary supersymmetric theory in the superfield formulation. In fact, all the secondary fields will be expressed in terms of basic fields (and auxiliary field B) due to the restrictions emerging from the application of horizontality condition (i.e. $\tilde{F}^{(2)} = F^{(2)}$), namely

$$\frac{1}{2}(dZ^M \wedge dZ^N) \tilde{F}_{MN} = \tilde{d}\tilde{A}^{(1)} \equiv dA^{(1)} = \frac{1}{2}(dx^\mu \wedge dx^\nu) F_{\mu\nu}, \quad (7)$$

where the super exterior derivative \tilde{d} and the connection super one-form $\tilde{A}^{(1)}$ are defined as

$$\begin{aligned} \tilde{d} &= dZ^M \partial_M = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}, \\ \tilde{A}^{(1)} &= dZ^M \tilde{A}_M \\ &= dx^\mu B_\mu(x, \theta, \bar{\theta}) + d\theta \bar{\mathcal{F}}(x, \theta, \bar{\theta}) + d\bar{\theta} \mathcal{F}(x, \theta, \bar{\theta}). \end{aligned} \quad (8)$$

To observe the impact of (7), let us first expand $\tilde{d}\tilde{A}^{(1)}$ as

$$\begin{aligned} \tilde{d}\tilde{A}^{(1)} &= (dx^\mu \wedge dx^\nu)(\partial_\mu B_\nu) - (d\theta \wedge d\bar{\theta})(\partial_\theta \bar{\mathcal{F}}) \\ &\quad + (dx^\mu \wedge d\bar{\theta})(\partial_\mu \bar{\mathcal{F}} - \partial_{\bar{\theta}} B_\mu) - (d\theta \wedge d\bar{\theta})(\partial_\theta \mathcal{F} + \partial_{\bar{\theta}} \bar{\mathcal{F}}) \\ &\quad + (dx^\mu \wedge d\theta)(\partial_\mu \mathcal{F} - \partial_\theta B_\mu) - (d\bar{\theta} \wedge d\theta)(\partial_{\bar{\theta}} \mathcal{F}). \end{aligned} \quad (9)$$

We shall apply now the horizontality condition (7) to obtain the nilpotent symmetry transformations (4) for the

gauge and (anti-) ghost fields. This is expected. It can be recalled that we have laid the emphasis on the role of the nilpotent ($d^2 = 0$) exterior derivative $d = dx^\mu \partial_\mu$ for the origin of the (anti-) BRST symmetry transformations which leave the $F_{\mu\nu}$ of the 2-form $F^{(2)} = dA^{(1)}$ invariant (cf. the discussion after (4)). It will be noted, furthermore, that the kinetic energy of the $U(1)$ gauge field is constructed from the 2-form $F^{(2)}$. In fact, the application of the horizontality condition yields [26]

$$\begin{aligned} R_\mu(x) &= \partial_\mu C(x), & \bar{R}_\mu(x) &= \partial_\mu \bar{C}(x), & s(x) &= \bar{s}(x) = 0, \\ S_\mu(x) &= \partial_\mu B(x), & B(x) + \bar{B}(x) &= 0, & \mathcal{B}(x) &= \bar{\mathcal{B}}(x) = 0. \end{aligned} \quad (10)$$

The insertion of all the above values in the expansion (6) yields

$$\begin{aligned} B_\mu^{(h)}(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta \partial_\mu \bar{C}(x) + \bar{\theta} \partial_\mu C(x) + i\theta \bar{\theta} \partial_\mu B(x), \\ \mathcal{F}^{(h)}(x, \theta, \bar{\theta}) &= C(x) - i\theta B(x), \\ \bar{\mathcal{F}}^{(h)}(x, \theta, \bar{\theta}) &= \bar{C}(x) + i\bar{\theta} B(x). \end{aligned} \quad (11)$$

This equation leads to the derivation of the (anti-) BRST symmetries for the gauge and (anti-) ghost fields of the Abelian gauge theory (cf. (4)). In addition, this exercise provides the physical interpretation for the (anti-) BRST charges $Q_{(a)b}$ as the generators (cf. (5)) of translations (i.e. $\lim_{\bar{\theta} \rightarrow 0} (\partial/\partial\theta)$, $\lim_{\theta \rightarrow 0} (\partial/\partial\bar{\theta})$) along the Grassmannian directions of the supermanifold. Both these observations can be succinctly expressed, in a combined fashion, by re-writing the super expansion (6) as

$$\begin{aligned} B_\mu^{(h)}(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta(s_{ab}A_\mu(x)) + \bar{\theta}(s_bA_\mu(x)) \\ &\quad + \theta\bar{\theta}(s_b s_{ab}A_\mu(x)), \\ \mathcal{F}^{(h)}(x, \theta, \bar{\theta}) &= C(x) + \theta(s_{ab}C(x)) + \bar{\theta}(s_bC(x)) \\ &\quad + \theta\bar{\theta}(s_b s_{ab}C(x)), \\ \bar{\mathcal{F}}^{(h)}(x, \theta, \bar{\theta}) &= \bar{C}(x) + \theta(s_{ab}\bar{C}(x)) + \bar{\theta}(s_b\bar{C}(x)) \\ &\quad + \theta\bar{\theta}(s_b s_{ab}\bar{C}(x)). \end{aligned} \quad (12)$$

In other words, after the application of the horizontality condition (7), we obtain the super 1-form connection $\tilde{A}_{(h)}^{(1)}$ (as $\tilde{A}_{(h)}^{(1)} = dx^\mu B_\mu^{(h)} + d\theta \bar{\mathcal{F}}^{(h)} + d\bar{\theta} \mathcal{F}^{(h)}$) such that $\tilde{d}\tilde{A}_{(h)}^{(1)} = dA$ is readily satisfied. It is clear from (11) that the horizontality condition enforces the fermionic superfields $(\bar{\mathcal{F}}(x, \theta, \bar{\theta}))\mathcal{F}(x, \theta, \bar{\theta})$ to become (anti-) chiral due to the equivalence between the translation generators operating on superfields of the supermanifold and the nilpotent symmetry transformations $s_{(a)b}$ acting on the local fields (cf. (5)) of the ordinary manifold.

4 Unique nilpotent symmetries for the complex scalar fields: Augmented superfield formalism with a gauge-invariant restriction

In this section, we derive the exact and unique nilpotent (anti-) BRST symmetry transformations for the complex

scalar fields in QED by exploiting a gauge-invariant restriction on the six (4, 2)-dimensional supermanifold. In this *gauge-invariant* restriction, once again, \tilde{d} and $\tilde{A}^{(h)}$ are going to play crucial roles. Thus, there is a mathematically beautiful interplay between the horizontality restriction and this new restriction. In fact, the new restriction turns out to be complementary in nature to the horizontality condition. To corroborate this assertion, let us begin with this new gauge-invariant restriction on the supermanifold

$$\Phi^*(x, \theta, \bar{\theta})(\tilde{d} + ie\tilde{A}_{(h)}^{(1)})\Phi(x, \theta, \bar{\theta}) = \phi^*(x)(d + ieA^{(1)})\phi(x), \quad (13)$$

where $\tilde{A}_{(h)}^{(1)} = dx^\mu B_\mu^{(h)} + d\theta \bar{\mathcal{F}}^{(h)} + d\bar{\theta} \mathcal{F}^{(h)}$ with superfield expansions for the multiplet superfields as quoted in (11) and the super expansion for the superfields $\Phi(x, \theta, \bar{\theta})$ and $\Phi^*(x, \theta, \bar{\theta})$, corresponding to the basic matter fields $\phi(x)$ and $\phi^*(x)$, are

$$\begin{aligned} \Phi(x, \theta, \bar{\theta}) &= \phi(x) + i\theta \bar{f}_1(x) + i\bar{\theta} f_2(x) + i\theta \bar{\theta} b(x), \\ \Phi^*(x, \theta, \bar{\theta}) &= \phi^*(x) + i\theta \bar{f}_2^*(x) + i\bar{\theta} f_1^*(x) + i\theta \bar{\theta} \bar{b}^*(x), \end{aligned} \quad (14)$$

where the number of fermionic secondary fields $\bar{f}_1(x)$, $f_1^*(x)$, $f_2(x)$, $\bar{f}_2^*(x)$ do match with the number of bosonic secondary fields $\phi(x)$, $\phi^*(x)$, $b(x)$, $\bar{b}^*(x)$ to maintain one of the basic requirements of a supersymmetric field theory. In the limit $(\theta, \bar{\theta}) \rightarrow 0$, we retrieve the local starting basic complex scalar fields $\phi(x)$ and $\phi^*(x)$. It is evident that the r.h.s. (i.e. $dx^\mu \phi^*(\partial_\mu + ieA_\mu)\phi$) of (13) is a $U(1)$ gauge-invariant term. The first term on the l.h.s. of (13) has the following expansion:

$$\begin{aligned} \Phi^*(x, \theta, \bar{\theta})\tilde{d}\Phi(x, \theta, \bar{\theta}) &= \Phi^*(x, \theta, \bar{\theta})(dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}) \\ &\quad \times \Phi(x, \theta, \bar{\theta}). \end{aligned} \quad (15)$$

It is straightforward to note that $\partial_\theta \Phi = i\bar{f}_1 + i\bar{\theta} b$, $\partial_{\bar{\theta}} \Phi = if_2 - i\theta b$ if we take into account the expansion (14) for Φ . The second term on the l.h.s. of (13) can be expressed as

$$\begin{aligned} \Phi^*(x, \theta, \bar{\theta})\tilde{A}_{(h)}^{(1)}\Phi(x, \theta, \bar{\theta}) &= \Phi^*(x, \theta, \bar{\theta})(dx^\mu B_\mu^{(h)} + d\theta \bar{\mathcal{F}}^{(h)} \\ &\quad + d\bar{\theta} \mathcal{F}^{(h)})\Phi(x, \theta, \bar{\theta}). \end{aligned} \quad (16)$$

It is clear that, from the above two equations, we shall obtain the coefficients of the differentials dx^μ , $d\theta$ and $d\bar{\theta}$. It is convenient algebraically to first focus on the coefficients of $d\theta$ and $d\bar{\theta}$ that emerge from (15) and (16). In explicit form, the former (i.e. (15)) leads to the following expressions in terms of the differentials $d\theta$ and $d\bar{\theta}$

$$d\theta \left[(i\phi^* \bar{f}_1) - \theta (\bar{f}_2^* \bar{f}_1) - \bar{\theta} (f_1^* \bar{f}_1 - i\phi^* b) + \theta \bar{\theta} (\bar{f}_2^* b - \bar{b}^* \bar{f}_1) \right], \quad (17)$$

$$d\bar{\theta} \left[(i\phi^* f_2) - \theta (f_2^* f_2 + i\phi^* b) - \bar{\theta} (f_1^* f_2) + \theta \bar{\theta} (f_1^* b - \bar{b}^* f_2) \right]. \quad (18)$$

The analogues of the above expressions that emerge from (16) are

$$\begin{aligned} & ied\theta \left[(\phi^* \bar{C}\phi) + i\theta (\bar{f}_2^* \bar{C}\phi - \phi^* \bar{C}\bar{f}_1) \right. \\ & \quad + i\bar{\theta} (\phi^* B\phi - \phi^* \bar{C}f_2 + f_1^* \bar{C}\phi) \\ & \quad + \theta\bar{\theta} \{ \bar{f}_2^* B\phi - \bar{f}_2^* \bar{C}f_2 + \bar{f}_1^* \bar{C}\bar{f}_1 + \phi^* B\bar{f}_1 \\ & \quad \left. + i(\bar{b}^* \bar{C}\phi + \phi^* \bar{C}b) \right], \end{aligned} \quad (19)$$

$$\begin{aligned} & ied\bar{\theta} \left[(\phi^* C\phi) + i\theta (\bar{f}_2^* C\phi - \phi^* C\bar{f}_1 - \phi^* B\phi) \right. \\ & \quad + i\bar{\theta} (f_1^* C\phi - \phi^* C f_2) \\ & \quad + \theta\bar{\theta} \{ \phi^* B f_2 - \bar{f}_2^* C f_2 + f_1^* C \bar{f}_1 + f_1^* B\phi \\ & \quad \left. + i(\bar{b}^* C\phi + \phi^* Cb) \right]. \end{aligned} \quad (20)$$

Finally, collecting the coefficients of $d\theta$ and $d\bar{\theta}$ from the above four equations, we obtain

$$\begin{aligned} & d\theta \left[i(\phi^* \bar{f}_1 + e\phi^* \bar{C}\phi) - \theta (\bar{f}_2^* \bar{f}_1 + e\bar{f}_2^* \bar{C}\phi - e\phi^* \bar{C}\bar{f}_1) \right. \\ & \quad + \bar{\theta} (i\phi^* b - f_1^* \bar{f}_1 + e\phi^* \bar{C}f_2 - e f_1^* \bar{C}\phi - e\phi^* B\phi) \\ & \quad + \theta\bar{\theta} [\bar{f}_2^* b - \bar{b}^* \bar{f}_1 + ie \{ \bar{f}_2^* B\phi - \bar{f}_2^* \bar{C}f_2 + \bar{f}_1^* \bar{C}\bar{f}_1 + \phi^* B\bar{f}_1 \\ & \quad \left. + i(\bar{b}^* \bar{C}\phi + \phi^* \bar{C}b) \} \right], \end{aligned} \quad (21)$$

$$\begin{aligned} & d\bar{\theta} \left[i(\phi^* f_2 + e\phi^* C\phi) \right. \\ & \quad - \theta (f_2^* f_2 + i\phi^* b + e f_2^* C\phi - e\phi^* C\bar{f}_1 - e\phi^* B\phi) \\ & \quad - \bar{\theta} (f_1^* f_2 - e\phi^* C f_2 + e f_1^* C\phi) + \theta\bar{\theta} [f_1^* b - \bar{b}^* f_2 \\ & \quad + ie \{ f_1^* B\phi - \bar{f}_2^* C f_2 + f_1^* C \bar{f}_1 + \phi^* B f_2 \\ & \quad \left. + i(\bar{b}^* C\phi + \phi^* Cb) \} \right]. \end{aligned} \quad (22)$$

Setting equal to zero the coefficients of $d\theta$, $d\theta(\theta)$, $d\theta(\bar{\theta})$ and $d\theta(\theta\bar{\theta})$ separately and independently, we obtain the following four relations (for $\phi^* \neq 0$):

$$\begin{aligned} & \bar{f}_1 = -e\bar{C}\phi, \quad \bar{C}\bar{f}_1 = 0, \quad b = -ie(B\phi - \bar{C}f_2), \\ & \bar{f}_2^* (b + ieB\phi - ie\bar{C}f_2) + (-\bar{b}^* + ie\bar{f}_1^* \bar{C} + ie\phi^* B)\bar{f}_1 \\ & - e(\bar{b}^* \bar{C}\phi + \phi^* \bar{C}b) = 0. \end{aligned} \quad (23)$$

In an exactly similar fashion, the equality of the coefficients of $d\bar{\theta}$, $d\bar{\theta}(\theta)$, $d\bar{\theta}(\bar{\theta})$ and $d\bar{\theta}(\theta\bar{\theta})$ to zero leads to the following relations (for $\phi^* \neq 0$):

$$\begin{aligned} & f_2 = -eC\phi, \quad b = -ie(B\phi + C\bar{f}_1), \quad C f_2 = 0, \\ & f_1^* (b + ieC\bar{f}_1 + ieB\phi) - \bar{b}^* (f_2 + eC\phi) \\ & + ie\phi^* B f_2 - e\phi^* Cb = 0. \end{aligned} \quad (24)$$

With $f_2 = -eC\phi$, $\bar{f}_1 = -e\bar{C}\phi$ as inputs, it is clear that (23) and (24) lead to $b = -ie(B + e\bar{C}C)\phi$. Furthermore, it is straightforward to note that $\bar{C}\bar{f}_1 = 0$ and $C f_2 = 0$ are automatically satisfied and the last entries of (23) and (24) are also consistent with the above values of \bar{f}_1 , f_2 and b . Thus, the independent relations that emerge from the comparison of the coefficients of $d\theta$ and $d\bar{\theta}$ of the l.h.s. and the r.h.s. of (13) are

$$\bar{f}_1 = -e\bar{C}\phi, \quad f_2 = -eC\phi, \quad b = -ie(B + e\bar{C}C)\phi, \quad (25)$$

which lead to the expansion of the superfield $\Phi(x, \theta, \bar{\theta})$, in terms of the (anti-) BRST transformations $s_{(a)b}$ of (4) for the scalar field $\phi(x)$, as follows:

$$\Phi(x, \theta, \bar{\theta}) = \phi(x) + \theta(s_{ab}\phi(x)) + \bar{\theta}(s_b\phi(x)) + \theta\bar{\theta}(s_b s_{ab}\phi(x)). \quad (26)$$

Now let us concentrate on the computation of the coefficients of dx^μ from the l.h.s. of (13). Written in an explicit form, these terms are

$$dx^\mu \left[\Phi^* \partial_\mu \Phi + ie\Phi^* B_\mu^{(h)} \Phi \right]. \quad (27)$$

The first term of the above equation contributes to the following:

$$\begin{aligned} & dx^\mu \left[(\phi^* \partial_\mu \phi) + i\theta (\phi^* \partial_\mu \bar{f}_1 + \bar{f}_2^* \partial_\mu \phi) + i\bar{\theta} (\phi^* \partial_\mu f_2 + f_1^* \partial_\mu \phi) \right. \\ & \quad \left. + i\theta\bar{\theta} (\phi^* \partial_\mu b + \bar{b}^* \partial_\mu \phi + i f_1^* \partial_\mu \bar{f}_1 - i \bar{f}_2^* \partial_\mu f_2) \right]. \end{aligned} \quad (28)$$

On the other hand, such a contribution coming from the second term is

$$dx^\mu \left[(ie\phi^* A_\mu \phi) - e\theta(K_\mu) - e\bar{\theta}(L_\mu) - e\theta\bar{\theta}(M_\mu) \right], \quad (29)$$

where the exact and explicit expressions for K_μ , L_μ and M_μ are

$$\begin{aligned} K_\mu &= \phi^* A_\mu \bar{f}_1 - i\phi^* \partial_\mu \bar{C}\phi + \bar{f}_2^* A_\mu \phi, \\ L_\mu &= \phi^* A_\mu f_2 - i\phi^* \partial_\mu C\phi + f_1^* A_\mu \phi, \\ M_\mu &= \phi^* A_\mu b + \phi^* \partial_\mu B\phi + \phi^* \partial_\mu C\bar{f}_1 - \phi^* \partial_\mu \bar{C}f_2 + \bar{b}^* A_\mu \phi \\ & \quad + f_1^* \partial_\mu \bar{C}\phi - \bar{f}_2^* \partial_\mu C\phi + i f_1^* A_\mu \bar{f}_1 - i \bar{f}_2^* A_\mu f_2. \end{aligned} \quad (30)$$

It is now evident that the coefficient of the pure differential dx^μ from the l.h.s. does match with that of the r.h.s. (i.e. $dx^\mu \phi^* (\partial_\mu + ieA_\mu)\phi$). Collecting the coefficients of $dx^\mu(\theta)$ and $dx^\mu(\bar{\theta})$ from (28), (29) and (30), we obtain the following expressions:

$$i\phi^* \partial_\mu \bar{f}_1 + i\bar{f}_2^* \partial_\mu \phi - e\bar{f}_2^* A_\mu \phi - e\phi^* A_\mu \bar{f}_1 + ie\phi^* \partial_\mu \bar{C}\phi, \quad (31)$$

$$i\phi^* \partial_\mu f_2 + i f_1^* \partial_\mu \phi - e f_1^* A_\mu \phi - e\phi^* A_\mu f_2 + ie\phi^* \partial_\mu C\phi. \quad (32)$$

Exploiting the inputs from (25) and setting equal to zero the above coefficients (31) and (32), we obtain the following relations:

$$i(\bar{f}_2^* - e\phi^* \bar{C})(D_\mu \phi) = 0, \quad i(f_1^* - e\phi^* C)(D_\mu \phi) = 0. \quad (33)$$

It is obvious from our interacting gauge system that $D_\mu \phi \neq 0$. Thus, we obtain the exact expressions for the secondary fields of the expansion in (14): $\bar{f}_2^* = e\phi^* \bar{C}$, $f_1^* = e\phi^* C$. The collection of the coefficients of $dx^\mu(\theta\bar{\theta})$ from (28), (29) and (30) yields

$$\begin{aligned} & i(\phi^* \partial_\mu b + \bar{b}^* \partial_\mu \phi) - f_1^* \partial_\mu \bar{f}_1 + \bar{f}_2^* \partial_\mu f_2 - e\phi^* A_\mu b \\ & \quad - e\phi^* \partial_\mu B\phi - e f_1^* \partial_\mu \bar{C}\phi + ie(\bar{f}_2^* A_\mu f_2 - f_1^* A_\mu \bar{f}_1) \\ & \quad - e\phi^* \partial_\mu C\bar{f}_1 + e\phi^* \partial_\mu \bar{C}f_2 + e\bar{f}_2^* \partial_\mu C\phi - e\bar{b}^* A_\mu \phi. \end{aligned} \quad (34)$$

The substitution of the values of the secondary fields $f_1^*, \bar{f}_2^*, b, \bar{f}_1, f_2$ in terms of the basic fields, in the above expression, finally leads to

$$i[\bar{b}^* - ie(B + eC\bar{C})\phi^*](D_\mu\phi), \quad (35)$$

which should be logically set equal to zero because there is no term corresponding to it on the r.h.s. of (13). Thus, we obtain the neat expression for \bar{b}^* : $\bar{b}^* = ie(B + eC\bar{C})\phi^*$ for $D_\mu\phi \neq 0$. This establishes the fact that all the secondary fields of the super expansion of $\Phi^*(x, \theta, \bar{\theta})$ can be expressed uniquely in terms of the basic and auxiliary fields due to the constraint (13) on the supermanifold. The insertion of these values in (14) leads to the following expansion of $\Phi^*(x, \theta, \bar{\theta})$ in terms of the transformations (4):

$$\begin{aligned} \Phi^*(x, \theta, \bar{\theta}) &= \phi^*(x) + \theta(s_{ab}\phi^*(x)) + \bar{\theta}(s_b\phi^*(x)) \\ &+ \theta\bar{\theta}(s_b s_{ab}\phi^*(x)). \end{aligned} \quad (36)$$

Let us begin with an alternative version of the gauge-invariant restriction (13) on the supermanifold. This restriction, in terms of \tilde{d} and $\tilde{A}_{(h)}^{(1)}$, can be expressed as follows:

$$\Phi(x, \theta, \bar{\theta})(\tilde{d} - ie\tilde{A}_{(h)}^{(1)})\Phi^*(x, \theta, \bar{\theta}) = \phi(x)(d - ieA^{(1)})\phi^*(x), \quad (37)$$

where the r.h.s. of the above equation contains a single differential dx^μ which can be explicitly written as: $dx^\mu\phi(\partial_\mu - ieA_\mu)\phi^*$. It is evident from the r.h.s (i.e. $dx^\mu[\phi(D_\mu\phi)^*]$) that the above restriction is really a *gauge-invariant* restriction. The first term ($\Phi\tilde{d}\Phi^*$) on the l.h.s. of (37) leads to the following expansion:

$$\Phi\tilde{d}\Phi^* = dx^\mu\Phi\partial_\mu\Phi^* + d\theta\Phi\partial_\theta\Phi^* + d\bar{\theta}\Phi\partial_{\bar{\theta}}\Phi^*, \quad (38)$$

where $\partial_\theta\Phi^* = if_2^* + i\bar{\theta}\bar{b}^*$, $\partial_{\bar{\theta}}\Phi^* = if_1^* - i\theta\bar{b}^*$. Collecting first the coefficients of $d\theta$ and $d\bar{\theta}$ from the above expression, we obtain

$$d\theta\left[(i\phi\bar{f}_2^*) - \theta(\bar{f}_1\bar{f}_2^*) - \bar{\theta}(f_2\bar{f}_2^* - i\phi\bar{b}^*) + \theta\bar{\theta}(\bar{f}_1\bar{b}^* - b\bar{f}_2^*)\right], \quad (39)$$

$$d\bar{\theta}\left[(i\phi f_1^*) - \theta(\bar{f}_1 f_1^* + i\phi\bar{b}^*) - \bar{\theta}(f_2 f_1^*) + \theta\bar{\theta}(f_2\bar{b}^* - b f_1^*)\right]. \quad (40)$$

The second term $-ie\Phi\tilde{A}_{(h)}^{(1)}\Phi^* = -ie\Phi(dx^\mu B_\mu^{(h)} + d\theta\bar{\mathcal{F}}^{(h)} + d\bar{\theta}\mathcal{F}^{(h)})\Phi^*$ of the l.h.s. of (37) yields the following coefficients of the differentials $d\theta$ and $d\bar{\theta}$:

$$\begin{aligned} &-ied\theta\left[(\phi\bar{C}\phi^*) + i\theta(\bar{f}_1\bar{C}\phi^* - \phi\bar{C}\bar{f}_2^*) \right. \\ &+ i\bar{\theta}(f_2\bar{C}\phi^* - \phi\bar{C}f_1^* + \phi B\phi^*) \\ &+ i\theta\bar{\theta}(b\bar{C}\phi^* + \phi\bar{C}\bar{b}^* - i\phi B\bar{f}_2^* + i\bar{f}_1\bar{C}f_1^* \\ &\left. - i\bar{f}_1 B\phi^* - if_2\bar{C}\bar{f}_2^*)\right], \end{aligned} \quad (41)$$

$$\begin{aligned} &-ied\bar{\theta}\left[(\phi C\phi^*) + i\theta(\bar{f}_1 C\phi^* - \phi C\bar{f}_2^* - \phi B\phi^*) \right. \\ &+ i\bar{\theta}(f_2 C\phi^* - \phi C f_1^*) \\ &+ i\theta\bar{\theta}(b C\phi^* + \phi C\bar{b}^* - i\phi B f_1^* + i\bar{f}_1 C f_1^* \\ &\left. - if_2 B\phi^* - if_2 C\bar{f}_2^*)\right], \end{aligned} \quad (42)$$

where explicit expressions for the superfields $\bar{\mathcal{F}}^{(h)}$ and $\mathcal{F}^{(h)}$ have been taken into account from (11). Setting equal to zero the coefficients of $d\theta$, $d\theta(\theta)$, $d\theta(\bar{\theta})$ and $d\theta(\theta\bar{\theta})$ from the above four expressions, we obtain the following relationships (for $\phi \neq 0$):

$$\begin{aligned} \bar{f}_2^* &= e\bar{C}\phi^*, \quad \bar{C}\bar{f}_2^* = 0, \quad \bar{b}^* = ie(B\phi^* - \bar{C}f_1^*), \\ (\bar{f}_1 + e\phi\bar{C})\bar{b}^* - ie^2 B\bar{C}\phi^* + ie\bar{f}_1(\bar{C}f_1^* - B\phi^*) &= 0. \end{aligned} \quad (43)$$

Similarly, equating the coefficients of $d\bar{\theta}$, $d\bar{\theta}(\theta)$, $d\bar{\theta}(\bar{\theta})$ and $d\bar{\theta}(\theta\bar{\theta})$ to zero yields (for $\phi \neq 0$)

$$\begin{aligned} f_1^* &= eC\phi^*, \quad \bar{b}^* = ie(B + eC\bar{C})\phi^*, \quad C f_1^* = 0, \\ (f_2 + e\phi C)\bar{b}^* - ie f_2(C\bar{f}_2^* + B\phi^*) - ie\phi B f_1^* &= 0, \end{aligned} \quad (44)$$

where, at some places, $f_1^* = eC\phi^*$, $\bar{f}_2^* = e\bar{C}\phi^*$ have already been used. Finally, we obtain the following independent relations⁴

$$f_1^* = eC\phi^*, \quad \bar{f}_2^* = e\bar{C}\phi^*, \quad \bar{b}^* = ie(B + eC\bar{C})\phi^*. \quad (45)$$

All the other relations in (43) and (44) are automatically satisfied. To compute the coefficients of dx^μ from the l.h.s. of the (37), we have to focus on $[dx^\mu(\Phi\partial_\mu\Phi^*)]$ and $ie[dx^\mu(\Phi B_\mu^{(h)}\Phi^*)]$. The former leads to the following expression:

$$\begin{aligned} dx^\mu &\left[(\phi\partial_\mu\phi^*) + i\theta(\phi\partial_\mu\bar{f}_2^* + \bar{f}_1\partial_\mu\phi^*) + i\bar{\theta}(\phi\partial_\mu f_1^* + f_2\partial_\mu\phi^*) \right. \\ &\left. + i\theta\bar{\theta}(\phi\partial_\mu\bar{b}^* + b\partial_\mu\phi^* + if_2\partial_\mu\bar{f}_2^* - i\bar{f}_1\partial_\mu f_1^*)\right], \end{aligned} \quad (46)$$

and the latter term yields

$$-iedx^\mu\left[(\phi A_\mu\phi^*) + i\theta(U_\mu) + i\bar{\theta}(V_\mu) + i\theta\bar{\theta}(W_\mu)\right], \quad (47)$$

where the explicit expressions for U_μ, V_μ and W_μ are as follows:

$$\begin{aligned} U_\mu &= \phi A_\mu\bar{f}_2^* - i\phi\partial_\mu\bar{C}\phi^* + \bar{f}_1 A_\mu\phi^*, \\ V_\mu &= \phi A_\mu f_1^* - i\phi\partial_\mu C\phi^* + f_2 A_\mu\phi^*, \\ W_\mu &= \phi A_\mu\bar{b}^* + \phi\partial_\mu B\phi^* + \phi\partial_\mu C\bar{f}_2^* - \phi\partial_\mu\bar{C}f_1^* + b A_\mu\phi^* \\ &+ f_2\partial_\mu\bar{C}\phi^* - \bar{f}_1\partial_\mu C\phi^* + if_2 A_\mu\bar{f}_2^* - i\bar{f}_1 A_\mu f_1^*. \end{aligned} \quad (48)$$

It is evident that when we collect the coefficient of the ‘‘pure’’ dx^μ from (46) and (47), it exactly matches with the r.h.s. (i.e. $dx^\mu\phi(D_\mu\phi)^*$). Setting the coefficients of $dx^\mu(\theta)$

⁴ It should be noted that exactly the same results, as quoted in (45), can be obtained from the covariant version (A.1) of the restriction (37) where the $d\theta$ and $d\bar{\theta}$ components lead to these derivations. However, the components $dx^\mu(\theta)$, $dx^\mu(\bar{\theta})$ and $dx^\mu(\theta\bar{\theta})$ from (A.1) lead to the result (i.e. $(D_\mu\phi)^* = 0$) which is found to be repugnant to the key requirement of the present interacting theory (QED) where $(D_\mu\phi)^* \neq 0$.

and $dx^\mu(\bar{\theta})$ from the l.h.s. of (37) equal to zero leads to the following equations:

$$i(\bar{f}_1 + e\phi\bar{C})(D_\mu\phi)^* = 0, \quad i(f_2 + e\phi C)(D_\mu\phi)^* = 0, \tag{49}$$

where we have used the inputs from (45). It is obvious from our present theory of QED that $(D_\mu\phi)^* \neq 0$. Thus, we obtain $\bar{f}_1 = -e\bar{C}\phi, f_2 = -eC\phi$ from (49). Finally, we set equal to zero the coefficient of $dx^\mu(\theta\bar{\theta})$ that emerges from (46), (47) and (48). We use in this computation the expressions given in (45) and the values of \bar{f}_1 and f_2 . Ultimately, we obtain the following equation:

$$[ib + e(B + e\bar{C}C)\phi](D_\mu\phi)^* = 0, \tag{50}$$

which leads to the derivation of b : $b = -ie(B + e\bar{C}C)\phi$ for $(D_\mu\phi)^* \neq 0$. Thus, we establish that the secondary fields of the expansion (14) can also be determined exactly and uniquely in terms of the basic and auxiliary fields of the theory if we exploit the gauge-invariant restriction (37) on the six (4, 2)-dimensional supermanifold. Finally, these values (either derived from (13) or (37)) lead to the expansion of the super matter fields as given in (26) and (36) in terms of off-shell nilpotent transformations $s_{(a)b}$ listed in (4).

5 Conclusions

In our present endeavour, we have exploited the gauge-invariant restrictions (cf. (13) and (37)) on the six (4, 2)-dimensional supermanifold to compute exactly and uniquely the off-shell nilpotent (anti-) BRST symmetry transformations (cf. (4)) for the complex scalar fields that are coupled to the 1-form $U(1)$ gauge field A_μ in a dynamically closed manner. The above gauge-invariant restrictions owe their origin to the (super) covariant derivatives defined on the supermanifolds. Thus, we have been able to provide a unique resolution to an outstanding problem in the context of the superfield approach to the BRST formalism. It is worthwhile to lay emphasis on the fact that the *covariant* versions (cf. (A.1) and the associated footnote) of the above *gauge-invariant* restrictions do not lead to the exact and acceptable derivation of the nilpotent (anti-) BRST symmetry transformations for the complex scalar fields of a 4D interacting $U(1)$ gauge theory in a logical fashion. This fact has been discussed in detail in Appendix.

We would like to lay stress on the fact that the usual horizontality condition $\tilde{F}^{(2)} = F^{(2)}$ (cf. (7)), responsible for the exact derivation of the nilpotent (anti-) BRST symmetry transformations for the gauge and (anti-) ghost fields, is basically a *covariant* restriction on the supermanifold. This is because of the fact that, for the non-Abelian gauge theory, the 2-form $F^{(2)}$ transforms as $F^{(2)} \rightarrow (F^{(2)})' = UF^{(2)}U^{-1}$, where U is the Lie group valued gauge transformation corresponding to the non-Abelian gauge theory under consideration (see. e.g. [6, 7] for details). It is merely an interesting coincidence that, for

the interacting $U(1)$ gauge theory (i.e. QED), the above covariant transformation of the 2-form $F^{(2)}$ reduces to a gauge-invariant transformation. It will be noted, however, that the derivation of the exact nilpotent (anti-) BRST symmetry transformations for the matter fields, depends *only* on the gauge-invariant restriction defined on the supermanifold and its covariant version leads to misleading results (cf. the appendix). This discrepancy is an important point in our whole discussion of the augmented superfield approach to the BRST formalism.

In our earlier works [25–31], we have proposed a consistent extension of the usual superfield formulation where, in addition to the horizontality condition, the restrictions emerging from the equality of the conserved quantities have been tapped on the supermanifold for the consistent derivation of the nilpotent symmetry transformations for the matter fields and other fields of the theory (see, e.g., [31] for details). However, these transformations for the matter (and other relevant) fields have *not* turned out to be unique. This is why our present work is important, in the sense that, we are able to derive *all* the nilpotent symmetry transformations *together* for the gauge, matter and (anti-) ghost fields in a unique manner. The restrictions in our present work are such that

- (i) they owe their origin to the (super) exterior derivatives (d) d and (super) 1-form connections ($\tilde{A}^{(1)}$ $A^{(1)}$,
- (ii) there is a mutual consistency and complementarity between these restrictions, in the sense that, the geometrical interpretations for $s_{(a)b}$ and $Q_{(a)b}$ remain intact, and
- (iii) they form the key ingredients of the theoretical arsenal of the *augmented* superfield approach to BRST formalism.

Our earlier works [25–32] and the present work are christened as the augmented superfield formalism because they turn out to be the consistent extensions, and in some sense generalizations, of the usual superfield approach to BRST formalism.

We have exploited the key ideas of the augmented superfield approach to the BRST formalism for the derivation of the unique nilpotent symmetry transformations for the Dirac fields in an interacting $U(1)$ gauge theory where the Abelian gauge field A_μ couples to the matter conserved current constructed by the Dirac fields alone [32]. A natural extension of our present work (and the earlier works [25–32]) is to check the validity of our proposal in the case of an interacting non-Abelian gauge theory [35] which is certainly a more general interacting system than the interacting Abelian gauge theories (i.e. QED). Furthermore, it would be a very nice endeavour to obtain the nilpotent symmetry transformations for *all* the fields of an interacting gauge theory by exploiting a single restriction on the supermanifold. We have been able to achieve that for the 4D interacting 1-form (non-) Abelian gauge theories by exploiting a gauge-invariant restriction that is found to owe its origin to a couple of covariant derivatives and their intimate connection with the curvature 2-form of the 1-form gauge fields [35–37]. It is worthwhile

to note that the usual superfield formalism has also been exploited in obtaining the nilpotent (anti-) BRST symmetries for the gauge and (anti-) ghost fields in the context of gravitational theories [18, 19]. It would be a very interesting venture to find out the usefulness of our proposal for the gravitational theories where matter fields (especially fermions) are in interaction with the gravitational (tetrad) fields. This issue is being intensively investigated at the moment and our results will be reported in our forthcoming future publications [38].

Acknowledgements. Stimulating conversations with L. Bonora (SISSA, Italy), K.S. Narain (AS-ICTP, Italy) and M. Tonin (Padova, Italy) are gratefully acknowledged. Thanks are also due to the referee for some very useful suggestions on the style and text of our present work.

Appendix

Let us begin with the following gauge covariant restriction on the six (4, 2)-dimensional supermanifold⁵:

$$(\tilde{d} - ie\tilde{A}_{(h)}^{(1)})\Phi^*(x, \theta, \bar{\theta}) = (d - ieA^{(1)})\phi^*(x), \quad (\text{A.1})$$

where the r.h.s. of the above equation is a single term (i.e. $dx^\mu[\partial_\mu\phi^*(x) - ieA_\mu\phi^*(x)]$) with the spacetime differential dx^μ alone and $\tilde{A}_{(h)}^{(1)} = dx^\mu B_\mu^{(h)} + d\theta\tilde{\mathcal{F}}^{(h)} + d\bar{\theta}\tilde{\mathcal{F}}^{(h)}$ is the super 1-form connection after the application of the horizontality condition (cf. (11)). The expanded version of the l.h.s., however, contains the differentials dx^μ , $d\theta$ and $d\bar{\theta}$ and their coefficients. In fact, the first term of the l.h.s. of (A.1) yields

$$\tilde{d}\Phi^*(x, \theta, \bar{\theta}) = dx^\mu\partial_\mu\Phi^* + d\theta\partial_\theta\Phi^* + d\bar{\theta}\partial_{\bar{\theta}}\Phi^*. \quad (\text{A.2})$$

It is clear from the expansion (14) that $\partial_\theta\Phi^* = i\bar{f}_2^* + i\bar{\theta}\bar{b}^*$ and $\partial_{\bar{\theta}}\Phi^* = if_1^* - i\theta\bar{b}^*$. The second term of the l.h.s. of (A.1) can be written as

$$-ie\tilde{A}_{(h)}^{(1)}\Phi^* = -iedx^\mu B_\mu^{(h)}\Phi^* - ied\theta\tilde{\mathcal{F}}^{(h)}\Phi^* - ied\bar{\theta}\tilde{\mathcal{F}}^{(h)}\Phi^*. \quad (\text{A.3})$$

It is evident from (A.2) and (A.3) that we shall have the coefficients of dx^μ , $d\theta$ and $d\bar{\theta}$ from both the terms of the l.h.s. of (A.1). Let us, first of all, focus on the coefficients of $d\theta$

⁵ There exists another analogous gauge covariant restriction $(\tilde{d} + ie\tilde{A}_{(h)}^{(1)})\Phi(x, \theta, \bar{\theta}) = (d + ieA^{(1)})\phi(x)$ on the six (4, 2)-dimensional supermanifold that leads to similar kinds of conclusions as drawn from (A.1). The computational steps for the former are exactly the same as that of the latter (i.e. (A.1)). In fact, as it turns out, in this restriction, different from (A.1), one obtains the unacceptable result which implies that $D_\mu\phi = 0$ for $e \neq 0, C \neq 0, \bar{C} \neq 0$. This is not the case, however, for the present QED under consideration.

and $d\bar{\theta}$. These are listed here:

$$d\theta \left[(i\bar{f}_2^* - ie\bar{C}\phi^*) - \theta(e\bar{C}\bar{f}_2^*) + \bar{\theta}(i\bar{b}^* + eB\phi^* - e\bar{C}f_1^*) + \theta\bar{\theta}(e\bar{C}\bar{b}^* - ieB\bar{f}_2^*) \right], \quad (\text{A.4})$$

$$d\bar{\theta} \left[(if_1^* - ieC\phi^*) - \theta(i\bar{b}^* + eB\phi^* + eC\bar{f}_2^*) - \bar{\theta}(eCf_1^*) + \theta\bar{\theta}(eC\bar{b}^* - ieBf_1^*) \right]. \quad (\text{A.5})$$

Setting equal to zero the coefficients of $d\theta$, $d\theta(\theta)$, $d\theta(\bar{\theta})$ and $d\theta(\theta\bar{\theta})$ separately and independently leads to the following relationships (for $e \neq 0$):

$$\begin{aligned} \bar{f}_2^* &= e\bar{C}\phi^*, & \bar{C}\bar{f}_2^* &= 0, \\ \bar{b}^* &= -ie[\bar{C}f_1^* - B\phi^*], & \bar{C}\bar{b}^* &= iB\bar{f}_2^*. \end{aligned} \quad (\text{A.6})$$

It is straightforward to check that the second entry and the fourth entry, in the above equation, are satisfied due to the first entry and the third entry, respectively. The equality of the coefficients of $d\theta$, $d\theta(\theta)$, $d\theta(\bar{\theta})$ and $d\theta(\theta\bar{\theta})$ to zero, leads (for $e \neq 0$) to

$$\begin{aligned} f_1^* &= eC\phi^*, & \bar{b}^* &= +ie[C\bar{f}_2^* + B\phi^*], \\ C\bar{f}_1^* &= 0, & C\bar{b}^* &= iBf_1^*. \end{aligned} \quad (\text{A.7})$$

Ultimately, (A.6) and (A.7) imply

$$f_1^* = eC\phi^*, \quad \bar{f}_2^* = e\bar{C}\phi^*, \quad \bar{b}^* = +ie[B + eC\bar{C}]\phi^*. \quad (\text{A.8})$$

Let us concentrate on the computation of the coefficients of dx^μ , $dx^\mu(\theta)$, $dx^\mu(\bar{\theta})$ and $dx^\mu(\theta\bar{\theta})$ that emerge from the l.h.s. of (A.1). It is elementary to check that

$$dx^\mu\partial_\mu\Phi^* = dx^\mu \left[\partial_\mu\phi^* + i\theta\partial_\mu\bar{f}_2^* + i\bar{\theta}\partial_\mu f_1^* + i\theta\bar{\theta}\partial_\mu\bar{b}^* \right]. \quad (\text{A.9})$$

The second term $-iedx^\mu(B_\mu^{(h)}\Phi^*)$ of the l.h.s. can be expanded as follows:

$$\begin{aligned} -iedx^\mu \left[A_\mu\phi^* + i\theta(A_\mu\bar{f}_2^* - i\partial_\mu\bar{C}\phi^*) + i\bar{\theta}(A_\mu f_1^* - i\partial_\mu C\phi^*) + i\theta\bar{\theta}(\partial_\mu B\phi^* + A_\mu\bar{b}^* + \partial_\mu C\bar{f}_2^* - \partial_\mu\bar{C}f_1^*) \right]. \end{aligned} \quad (\text{A.10})$$

It is quite obvious that the coefficient of the ‘‘pure’’ dx^μ of the l.h.s. matches with that of the r.h.s. in (A.1). Setting equal to zero the coefficients of $dx^\mu\theta$, $dx^\mu\bar{\theta}$ and $dx^\mu(\theta\bar{\theta})$ leads to

$$\begin{aligned} i\partial_\mu\bar{f}_2^* + eA_\mu\bar{f}_2^* - ie\partial_\mu\bar{C}\phi^* &= 0, \\ i\partial_\mu f_1^* + eA_\mu f_1^* - ie\partial_\mu C\phi^* &= 0, \\ i\partial_\mu\bar{b}^* + e(\partial_\mu B\phi^* + A_\mu\bar{b}^* + \partial_\mu C\bar{f}_2^* - \partial_\mu\bar{C}f_1^*) &= 0. \end{aligned} \quad (\text{A.11})$$

Inserting the values of f_1^* , \bar{f}_2^* and \bar{b}^* in the above from (A.8), we obtain

$$\begin{aligned} ie\bar{C}(D_\mu\phi)^* &= 0, & ieC(D_\mu\phi)^* &= 0, \\ -e(B + eC\bar{C})(D_\mu\phi)^* &= 0. \end{aligned} \quad (\text{A.12})$$

The above conditions lead to the absurd result that $(D_\mu\phi)^* = 0$ for $e \neq 0, C \neq 0, \bar{C} \neq 0$. One cannot choose $B = -eC\bar{C}$ in the last condition of (A.12) because that would lead to the condition that $\bar{b}^* = 0$. This is *not* the case as can be seen from the expansion (36).

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